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Consensus in a Multi-Agent System with Switching Topology

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Abstract

This article explores the mathematical framework for developing consensus algorithms in multi-agent systems, using both fixed and switching communication graphs. Consensus refers to the agreement among agents achieved by sharing local information. Local interactions realize this global objective, a key issue in multi-agent control, also known as cooperative control. The consensus equation can be formulated in either continuous or discrete time domains. This article focuses on deriving the consensus equation in the discrete time domain using Perron-Frobenius theory. The discrete time consensus equation is dependent upon the underline structure of the communication graph. For achieving consensus, two types of communication graphs are considered: fixed communication graphs and switching communication graphs. Consensus values for switching communication graphs and fixed communication graphs are derived for random and fixed initial state information of agents. The convergence of the consensus algorithm depends upon the eigenstructure of the Frobenius matrix, and it is constructed for fixed and switch communication graphs. The eigenvalues of the Frobenius matrix lie within the unit circle, so the trajectory of state information of each agent is exponentially stable and converges to a common value known as the consensus value at steady state. The consensus value for fixed and switching graphs is the average of their initial state information, but the time required for convergence of the algorithm in the case of switching graphs is greater than that for fixed communication graphs. This theoretical finding is illustrated via simulations.

Keywords: Multiple agent system (MAS), consensus, graph Laplacian, Frobenius matrix and algebraic graph theory

INTRODUCTION

For two decades, there has been active study in the field of multi-agent system (MAS) control [1]. A distributed technique and a centralized approach are typically used to control MAS systems.

The distributed strategy is increasingly favored due to physical constraints such as limited wireless communication range, restricted sensing areas, narrow bandwidth, large vehicle sizes, and the complex dynamics of each agent [2]. Managing and controlling these factors is challenging, which is why distributed control in multi-agent systems (MAS) is a promising research area. Several survey papers [3], [4], and [5] detail recent advancements in MAS coordination and control. Consensus is a crucial issue in MAS, and various information flow restrictions have been studied to explore different consensus techniques [6], [7], [8], [9], and [10]. Additionally, monographs [11], [12], and [13] offer

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recent reviews and progress reports. This article focuses on designing a consensus algorithm for MAS with both fixed and switching topologies using discrete Perron-Frobenius theory and includes a convergence analysis comparing consensus algorithms for switching versus fixed topologies.

Notations and Symbols

Matrices are denoted by capital, boldface letters, while their elements are represented by lowercase letters. $R^{n \times n}$ indicates a matrix of square dimension with real entries. All N agents are interconnected to each other using a specific structure known as network topology. Matrix F is known as Frobenius matrix, and ʎ, 1, w_1 are respectively, symbols used for eigenvalues right eigen vector and left eigen vector of matrix F. The matrices F, D, and A, respectively, stand for the Frobenius matrix, the diagonal matrix, and the adjacency matrix.

The article is arranged as follows: Section II describes algebraic graph theory as a prerequisite tool for designing consensus algorithms. Formulation of problem is explained in Section III. Section IV describes the simulation result, and the concluding remarks are explained in section V.

ALGEBRIAC GRAPH THEORY

To represent the MAS in mathematical form, a state space equation is required. Graph theory and matrix theory, along with the state space model, are the primary tools used in designing cooperative control of MAS.

Graph Theory

A communication graph (GN) is a representation of a set of agents in mathematics, specifically in graph theory, where specific pairs of agents are linked together by links. Mathematical abstractions known as vertices and edges, which connect certain pairings of vertices, are used to depict connected agents. Either directed or undirected edges are possible. A graph's neighbors are its vertices that are directly related to one another; these are denoted mathematically by the symbol Ni for agent i. A path connecting every pair of nodes in a graph is said to be linked. A linked diagram has no accessible vertices. A graph is an algebraically defined structure used to model a network topology in a network of interconnected agents, and it is given by G_N = (V, E). A set by agents is denoted by V = {$v_{1,} v_{2, \dots,}$ v_N}. Furthermore, the set of edges E denotes the relationships amongst agents, and it is represented as E = ($v_{i\,,}$ v_j) [13]. The spanning tree is another key idea in graph theory. We refer to a directed spanning tree when there is at least one corner point in a network with a route that is directed to every other agent. The direction of information flow from one agent to another, indicated by the edges, allows the diagram to be classified. One can have directed or undirected graphs.

Directed graph: A directed graph is characterized by edges that have a certain direction. Nodes are connected by edges, and certain paths cannot be followed back to the starting node. Arrows on the edges of directional diagrams indicate the direction of information flow. The source node needs to discover a different path back to the start node; it can only utilize the address provided to get to the destination node. A directed graph may occasionally experience issues with the point of no return. A communication network obstruction may result from this. Moreover, a directed graph is defined as a series of directed edges in the form (v_1, v_2), (v_2, v_3),... (vN_{-1}, vN), where vi belongs to V.

Undirected graph: An undirected graph is one where the edges lack a specified direction. In such a graph, communication links permit information to flow in both directions. In the case of undirected graphs, $A = A^T$, here matrix A is known as the connectivity matrix, and consequently, the matrix is symmetric about the diagonal. The edge of a graph is represented by an ordered pair, which may be expressed as (v_i, v_j). This only indicates that agent i immediately connects to agent j and communicates its information to it. An undirected graph (v_i, v_j). indicates that the two agents are interconnected and can exchange information in both directions. It is noteworthy to emphasize that in an undirected communication graph, redundant information from the same node from different sources overloads the communication channel, yet the information is always available when needed.

Graph Matrices

Graph matrices play a major role in consensus convergence for networked multi-agent systems, in addition to graph theory. A matrix is referred to as a non-negative matrix when all its entries are positive. In the same way, if every element in the vector is positive, the vector's shape is also not negative. The degree matrix is among the most crucial matrices in the construction of consensus methods. The matrix of degrees D = [d_{ij}] of a graph G_N is essentially a diagonal matrix with a degree of vertex, or the quantity of agents that are close to i. An adjacency matrix is a matrix that offers details on the connections between its constituent parts. For a graph G_N its mathematical adjacency matrix is represented by A = [a_{ij}] and is given by

$$A = \{ 1, if\ (vi, vj)\ \in\ E \quad 0, Otherwise$$

Matrix I is the identity matrix, and it has dimensions M x N and M = N. The size of the matrix depends on the number of agents present in a graph. The matrix F is stochastic. Stochastic matrices have an important role in the study of graphs. We say a matrix F is nonnegative if F ≥ 0 if all its elements are nonnegative. The matrix F is positive, F > 0, if it is all elements are strictly positive. If every row sum in a matrix F equals 1, then the matrix is row stochastic. If all the sums of the rows and columns of a matrix F equal 1, then it is doubly stochastic. Two row stochastic matrices E and F have a row stochastic product because E F 1 = E 1 = 1. A stochastic matrix's greatest eigenvalue is 1, and if matrix F ≥ 0 is row stochastic if and only if 1 is an eigenvector for the eigenvalue 1. Let square n x n matrix F ≥ 0 have all row sums equal to a constant c > 0 then

- ρ(F) = c, and it is an eigenvalue of F with eigenvector 1.
- If diagonal elements of matrix F are positive, i.e., $f_{ii} > 0$ for all i, then |λ| < c for all eigenvalues λ ≠ c.

Let A be the adjacency matrix of a graph G, then

- $\lambda_1 = \rho(A) = c$ is a simple eigenvalue if and only if A. Then *rank*(A)=N-1.
- If A has a spanning tree and if it is all diagonal elements, i.e., $a_{ii} > 0$ for all i, then $\lambda_1 = \rho(A) = c$ is unique.

FORMULATION OF PROBLEM

The mathematical formulation of the discrete time consensus algorithm is described in this section. Every agent i in a MAS is equipped with the discrete time state space equation shown below

$$x_i(k+1) = x_i(k) + \mu_i(k) \tag{1}$$

$x_i(k)$ and $x_i(k+1)$ is the state of i^th agent at k^th and at (k + 1)^th time instance respectively. $\mu_i(k)$ is the local control input of i^th agent and also $x_i(k), x_i(k+1)$ and$\mu_i(k) \in R$. In section III-A, the Perron discrete time system is used to obtain the discrete time consensus equation.

A Discrete-Time Consensus Control Protocol with Normalized Control

Suppose that agent i's normalized control input is as follows:

$$\mu_i(k) = \frac{1}{1+d_i} \sum_{j \in N_i} a_{ij}\, [x_j(k) - x_j(k)] \tag{2}$$

where d_i stands for the in-degree of agent i and a_{ij} stands for components of matrix A. When Equation (2) is substituted in Equation (1), we obtain

$$x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \sum_{j \in N_i} a_{ij} \left[x_j(k) - x_j(k)\right] \tag{3}$$

$$x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \left(-x_i(k) \sum_{j \in N_i} a_{ij} + \sum_{j \in N_i} a_{ij} x_j(k)\right) \tag{4}$$

From [13], $\sum_{j \in N_i} a_{ij} = d_i$ and if j = 1, 2, …., N then Equation (4) becomes

$$x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \left(-x_i(k) d_i + [a_{i1}, \cdots, a_{iN}]\, [x_1(k) \ \vdots\ x_N(k)]\right) \tag{5}$$

Equation (5) is an i^{th} agent discrete state space equation that is globally transformed into a matrix form as

$$x(k+1) = Ix(k) + \frac{I}{I+D}\left(-x(k)D + Ax(k)\right) \quad (6)$$

The identity matrix I in this case is equal to 1 in a scalar and further simplifying Equation (6) we get

$$x(k+1) = \left[I + \frac{I}{I+D}\left(-(D-A)\right)\right]x(k) \quad (7)$$

Simplifying Equation (7), we obtained consensus equation in discrete time domain as

$$x(k+1) = (I+D)^{-1}(I+A)x(k) \equiv Fx(k) \quad (8)$$

Corresponding discrete time consensus equation of switching communication graph can easily be derived from Equation (8) as

$$x(k+1) = (I+D)^{-1}(I+A)x(k) \equiv F(k)x(k) \quad (9)$$

Here x(k) and x(k + 1) ∈ R^N. F and F(k) is known as the Frobenius matrices for fixed and switching communication graphs. Matrix F and F(k) have N eigenvalues out of which one eigenvalue lies on the unit circle and N –1 eigenvalues mapped inside the unit circle of the complex z-plane. According to the Gershgorin circle theorem, N –1 eigenvalues of matrix F are found inside the unit circle; see the shaded area of Figure 1. One eigenvalue on the unit circle indicates that the discrete system given by equations (8) and (9) is Type-1 and moderately stable, and the state of all agents approaches a steady state value as time reaches infinity [13].

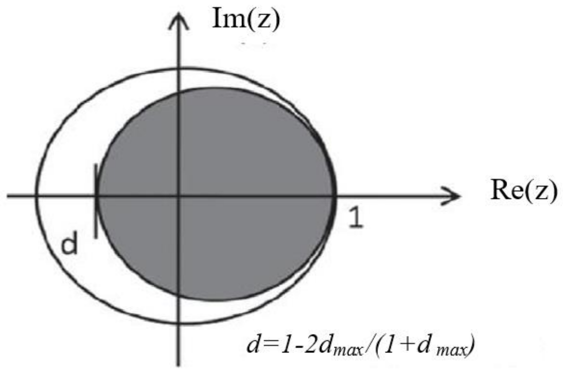

Figure 1. Region of eigenvalues of Frobenius matrix F.

F is a row stochastic matrix because F has row sum is equal to one; hence, we write F***1*** = ***1,*** where ***1*** is the right eigenvector of matrix F associated with eigenvalue $\lambda_1 = 1$.

B Analysis of Steady States

Steady state convergence of the discrete time consensus algorithm is described in this section. Assume that $w_1 = [p_1, p_2, \ldots\ldots, p_N]$ is the N-dimensional left eigenvector of the matrix F for $\lambda_1 = 1$. When the system achieves a steady state, then we have $x_{ss} = F\, x_{ss}$. Equation (8) converges to a consensus value c > 0 if network topology consists of a directed spanning tree [13]. Also, if w_1 is multiplied by both sides of the Equation (8) and $w_1^T F = w_1^T$ then we have

$$w_1^T x(k+1) = w_1^T Fx(k) = w_1^T x(k) = \sum_i \quad p_i x_i \quad (10)$$

The quantity $\sum_i \quad p_i x_i$ is not time variant for all time instances k. Therefore, initial state information of all agents asymptotically reached consensus value c > 0 and it is given as,

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$$C = \frac{\sum_{i=1}^{N} \; p_i x_i(0)}{\sum_{i=0}^{N} \; p_i} \tag{11}$$

SIMULATION RESULT

Consider a fixed communication graph shown in Figure 2 and consider all edge weight a_{ij} = 1. Then adjacency matrix A, diagonal matrix D, and identity matrix I are given by

$$A = (\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ )\ D = (\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ )\ I = (\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ) \tag{12}$$

Finally, Frobenius matrix F for the Figure 2 is given by

$$F = (I + D)^{-1}\ (I + A) = \ (0.5\ 0.5\ 0\ 0\ 0.5\ 0.5\ 0.5\ 0\ 0.5) \tag{13}$$

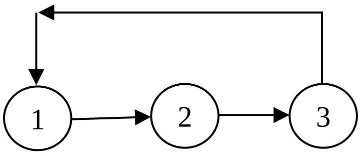

Figure 2. Fixed communication graph G.

Consider switching communication graphs as G_1, G_2, and G_3 shown in Figure 3, Figure 4, and Figure 5, respectively.

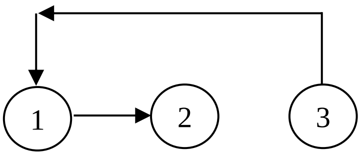

Figure 3. Graph G1.

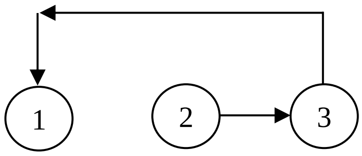

Figure 4. Graph G2.

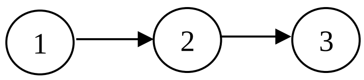

Figure 5. Graph G3.

Then adjacency matrix A, diagonal matrix D, and identity matrix I for graph G_1 is given by

$$A = (\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0) \quad D = (\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1) \quad I = (\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1) \tag{14}$$

and corresponding Frobenius matrix F_1 for the Figure 3 is given by

$$F_1 = (0.5\ 0.5\ 0\ 0\ 1\ 0\ 0.5\ 0\ 0.5). \tag{15}$$

Then adjacency matrix A, diagonal matrix D, and identity matrix I for graph G_2 is given by

$$A = (\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0)\ D = (\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1)\ I = (\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1) \tag{16}$$

and corresponding Frobenius matrix F_2 is given by

$$F_2 = (1\ 0\ 0\ 0\ 0.5\ 0.5\ 0.5\ 0\ 0.5). \tag{17}$$

Then adjacency matrix A, diagonal matrix D, and identity matrix I for graph G_3 is given by

$$A = (\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ )\ D = (\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ )\ I = (\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1). \tag{18}$$

and corresponding Frobenius matrix F_3 is given by

$$F_3 = (1\ 0\ 0\ 0\ 0.5\ 0.5\ 0.5\ 0\ 0.5). \tag{19}$$

Eigen values for Frobenius matrix F are $\lambda_{12} = 0.25 \pm 0.433i$ and $\lambda_3 = 1$, and corresponding left eigenvectors elements for Frobenius matrix F are p1 = 1, p2 = 0 and p3 = 0. Here initial state information for agents is assumed to be $x_i(0) = [0.811584, -0.746026, 0.629447]$ for agent i = 1, 2 and 3 respectively, using equation (11) consensus value determined as 0.21 which is achieved in 6 seconds and same is plotted in Figure 6. However, for switching communication graph Frobenius matrices are F_1, F_2 and F_3. For F_1 the eigen values are $\lambda_{12} = 0.5$ and $\lambda_3 = 1$ and corresponding eigen vectors are $p_i = \{0, 1, 0\}$. For F_2 the eigen values are $\lambda_{12} = 0.5$ and $\lambda_3 = 1$ and corresponding eigen vectors are $p_i = \{1, 0, 0\}$. For F_3 the eigen values are $\lambda_{12} = 0.5$ and $\lambda_3 = 1$ and corresponding eigen vectors are $p_i = \{0, 0, 1\}$. This structure G_1, G_2 and G_3 repeats 21 times then using Equation (11) consensus value determined as 0.31 which is achieved in 10 seconds and same is plotted in Figure 7. Considering agent number as initial state information for fixed and switching communication graph as $x_1(0) = 1$, $x_2(0) = 2$ and $x_3(0) = 3$ then for fixed communication graph consensus value is 2 and it is achieved in 10 seconds. For switching communication graph consensus value is 1.76 and it is achieved in 15 seconds. Figures 8 and 9 show consensus convergence for fixed initial state information $x_i = \{1, 2, 3\}$.

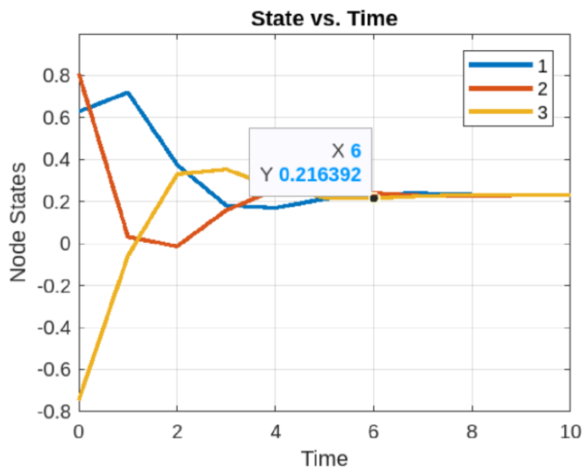

Figure 6. Plot for G.

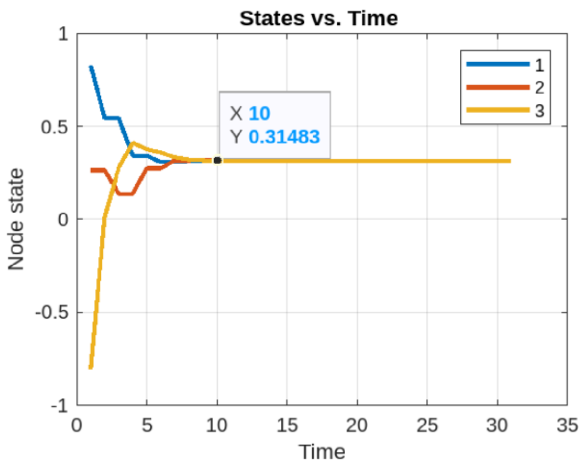

Figure 7. Plot for G_1, G_2 and G_3.

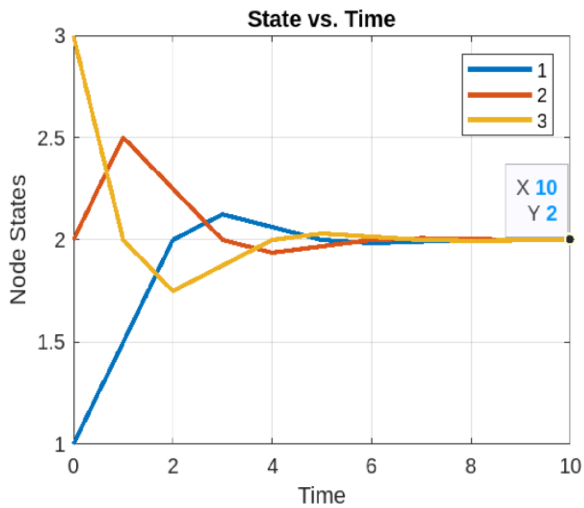

Figure 8. Plot for G.

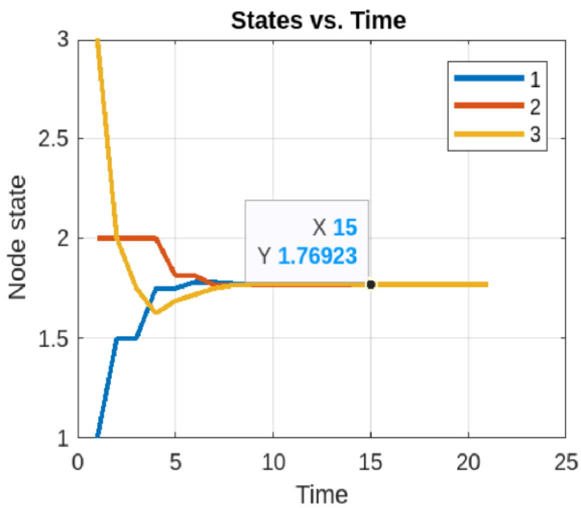

Figure 9. Plot for G_1, G_2 and G_3.

However, the simulation results displayed in (Figure 6) show that the state information of all agents converges to a consensus value of 0.21 in 6 seconds. In contrast, (Figure 7) demonstrates that the state information of all agents converges to a consensus value of 0.31 in 10 seconds. (Figure 8) shows that the consensus value reaches 2 in 10 seconds, while (Figure 9) indicates convergence to a consensus value of 1.76 in 15 seconds. Comparing Figures 6 through 9, it is evident that the consensus value for switching topologies is achieved more slowly than for fixed topologies. This delay is attributed to link failures in graphs G1, G2, and G3, as referenced in (Figures 3, 4, and 5). Random initial state information of each agent selected in between the range of +1 to -1 and fixed initial state information is considered as $X_i = \{1, 2, 3\}$. Table 1 shows a convergence comparison between fixed topology and switching topology. Clearly, convergence time for switching topology is larger than fixed topology.

Table 1. Comparative analysis of algorithm convergence.

Type of Topology	Convergence Time	Consensus Value
Fixed topology	6 Sec	0.21
Switching topology	10 Sec	0.31

CONCLUSION

In conclusion, this article provides a detailed mathematical framework for consensus algorithms in multi-agent systems, focusing on both fixed and switching communication graphs in the discrete time domain. The study shows that consensus, achieved through local interactions among agents, is influenced by the structure of the communication graph. Using Perron-Frobenius theory, the consensus equations were derived, and it was demonstrated that convergence is governed by the eigenstructure of the Frobenius matrix. The eigenvalues lie within the unit circle, ensuring exponential stability and convergence to a common consensus value. Although the consensus value is the average of the initial states of the agents in both fixed and switching graphs, the time required for convergence is longer for the switching graph. These theoretical findings were supported by simulations.

Declaration of Interest

There is no conflict of interest regarding the publication of this manuscript.

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